UDC 531.31

ON THE POSSIBLE TYPES OF CRITICAL CASES FOR LAGRANGE EQUATIONS OF SECOND KIND

PMM Vol. 36, №3, 1972, pp. 390-395

A. N. VEISSENBERG
(Yaroslavl')

(Received September 22, 1971)

We consider the types of critical cases arising in the general equations of a holonomic scleronomous system in independent coordinates. We examine the system's first-approximation matrix and we study the elementary divisors corresponding to this matrix. We prove a theorem on the stability of the trivial solution in one specific critical case when we use a function which is sign-definite in a part of the variables. After Liapunov's original work 1, 2 the critical cases in the general problem of stability of motion were considered in [3]. The algebraic unsolvability of stability problems in sufficiently complex critical cases was pointed out in [4].

1. Suppose that we are given the general equations of motion on a holonomic scleronomous system in independent coordinates

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = Q_i \qquad (i = 1, \ldots, n) \tag{1.1}$$

The system's kinetic energy is $T = (q^{\cdot})' A q^{\cdot} / 2 + (q^{\cdot})' A\ (q)\ q^{\cdot}$, where A is a constant positive-definite matrix $(A > 0)$. The elements of the matrix $A(q)$ are analytic in the components of vector $q, A\ (0) = 0$. The prime denotes the transpose. Let $q = q^{\cdot} = 0$ be the equilibrium position. By assuming the generalized forces Q_i to be stationary, system (1.1) can be rewritten as [5]

$$dx/dt = y, \qquad dy/dt = Qx + Ly + \nu\ (x,\ y) \tag{1.2}$$
$$(q = x,\ q^{\cdot} = y)$$

where Q, L are constant matrices; the components of the vector $\nu\ (x,\ y)$ are analytic and of not lower than second order.

The first-approximation is

$$P = \begin{Vmatrix} 0 & E \\ Q & L \end{Vmatrix} \tag{1.3}$$

where E is the unit matrix. The matrix P is of even order. We investigate the possibility of appearance in the spectrum $\sigma\ (P)$ of matrix P of zeros or of pure imaginary numbers, depending on the properties of the matrices Q, L. We study the corresponding types of elementary divisors. Without loss of generality the matrix Q (or L) is taken as having been reduced to a canonic Jordan form. Using the matrix equality

$$\begin{Vmatrix} P - \varkappa E \end{Vmatrix} \cdot \begin{Vmatrix} E & E \\ 0 & \varkappa E \end{Vmatrix} = \begin{Vmatrix} -\varkappa E & 0 \\ Q & Q + \varkappa L - \varkappa^2 E \end{Vmatrix}$$

we can obtain the matrix's characteristic polynomial $f\ (\varkappa)$

$$f(\varkappa) = (-1)^n \det \| Q + \varkappa L - \varkappa^2 E \| \tag{1.4}$$

After simple manipulations [6] the matrix $P - \varkappa E$ takes the form

$$\begin{Vmatrix} E & 0 \\ 0 & Q + \varkappa L - \varkappa^2 E \end{Vmatrix} \tag{1.5}$$

On the complex plane we consider the sets

$$\Theta = \{z : \operatorname{Im} z = 0, \ -\infty < \operatorname{Re} z \leqslant 0\}$$
$$\Lambda = \{z : \operatorname{Re} z \leqslant 0\}, \ \Omega = \{z : \operatorname{Re} z = 0\}$$

Theorem 1. 1. The number of zero eigenvalues of matrix P is not less than the number of elementary divisors of the matrix $Q - \lambda E$, corresponding to the zero eigenvalues of the matrix Q.

2. Suppose $Q = 0$, $L \neq 0$ and let $\lambda^l, \ldots, \lambda^k$ (g times), $(\lambda - \lambda_1)^{p_1}, \ldots, (\lambda - \lambda_r)^{p_r}$ (r times) be the set of elementary divisors of the matrix $L - \lambda E$ $(l + \ldots + k = m,$ $m + p_1 + \ldots + p_r = n)$. Then the elementary divisors of the matrix $P - \varkappa E$ are $\varkappa, \ldots, \varkappa$ ($n - g$ times), $\varkappa^{l+1}, \ldots, \varkappa^{k+1}$ (g times), $(\varkappa - \lambda_1)^{p_1}, \ldots, (\varkappa - \lambda_r)^{p_r}$ (r times).

3. Suppose $Q \neq 0$, $L = 0$, $\sigma(Q) \cap \Theta = \Theta$ and let $\lambda^l, \ldots, \lambda^k$ (g times), $(\lambda - \lambda_1)^{p_1}, \ldots, (\lambda - \lambda_r)^{p_r}$ (r times) be the set of elementary divisors of the matrix $Q - \lambda E$ $(l + \ldots + k = m, m + p_1 + \ldots + p_r = n)$. Then the elementary divisors of the matrix $P - \varkappa E$ are $\varkappa^{2l}, \ldots, \varkappa^{2k}$ (g times), $(\varkappa + i\sqrt{+\lambda_1})^{p_1}$, $(\varkappa - i\sqrt{-\lambda_1})^{p_1}, \ldots,$ $(\varkappa + i\sqrt{-\lambda r})^{p_r}$, $(\varkappa - i\sqrt{-\lambda_r})^{p_r}$ ($2r$ times).

4. Suppose $Q = L = 0$. Then the elementary divisors of the matrix $P - \varkappa E$ are $\varkappa^2, \ldots, \varkappa^2$ (n times).

Proof. 1. The matrix Q is considered reduced to a Jordan form. We examine the equality

$$Q + \varkappa L - \varkappa^2 E = \begin{Vmatrix} P_{11}(\varkappa) & P_{12}(\varkappa) \\ P_{21}(\varkappa) & P_{22}(\varkappa) \end{Vmatrix} \tag{1.6}$$

where the square matrices $P_{11}(\varkappa)$, $P_{22}(\varkappa)$ correspond, respectively, to the elementary divisors $\lambda^l, \ldots, \lambda^k$ (g times), and $(\lambda - \lambda_1)^{p_1}, \ldots, (\lambda - \lambda_r)^{p_r}$ (r times) of the matrix $Q - \lambda E$. We use the relation

$$(-1)^n f(\varkappa) = \det \| Q + \varkappa L - \varkappa^2 E \| = \sum_{\pi} (\operatorname{sgn} \pi) \alpha_{\pi(1),1}(\varkappa) \ldots \alpha_{\pi(n),n}(\varkappa) \tag{1.7}$$

where the summation extends over all permutations π of the set of all permutations of the integers from one to n, where $\alpha_{ij}(\varkappa)$ is the element of the matrix $\| Q + \varkappa L - \varkappa^2 E \|$ at the intersection of the i-th row and the j-th column. From (1.7) and from the form of the matrices $P_{11}(\varkappa)$, $P_{22}(\varkappa)$ it follows that the polynomial $f(\varkappa)$ does not contain terms with $\varkappa$ to a power less than g.

2. The matrix L is assumed reduced to a Jordan form. We examine the matrix (1.5) under the condition $Q = 0$. Considering [7] we obtain the desired set of elementary divisors after a union of the elementary divisors of $k \times k$ and $p_r \times p_r$ matrices of the type

$$\begin{Vmatrix} -\varkappa^2 & \varkappa & \ldots & 0 \\ 0 & -\varkappa^2 & \ldots & 0 \\ \ldots & & \ldots & \\ 0 & 0 & \ldots & \varkappa \\ 0 & 0 & \ldots & -\varkappa^2 \end{Vmatrix}, \quad \begin{Vmatrix} \varkappa\lambda_r - \varkappa^2 & \varkappa & \ldots & 0 \\ 0 & \varkappa\lambda_r - \varkappa^2 & \ldots & 0 \\ \ldots & & \ldots & \\ 0 & 0 & \ldots & \varkappa \\ 0 & 0 & \ldots & \varkappa\lambda_r - \varkappa^2 \end{Vmatrix}$$

It is clear that $\varkappa, \ldots, \varkappa$ ($k-1$ times), $\varkappa^{k+1}$ form the set of elementary divisors of the first matrix; $\varkappa, \ldots, \varkappa$ (p_r times), $(\varkappa-\lambda_r)^{p_r}$ form the analogous set for the other matrix. In system (1.3) a critical case is possible only when $\sigma(L) \in \Lambda$.

3. The matrix Q is assumed reduced to a Jordan form. We examine the matrix (1.5) under the condition $L=0$. We obtain the desired set of elementary divisors after a union of the elementary divisors of $k \times k$ and $p_r \times p_r$ matrices of the type

$$\left\|\begin{array}{cccc} -\varkappa^2 & 1 & \ldots & 0 \\ 0 & -\varkappa^2 & \ldots & 0 \\ \ldots & & \ldots & \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & -\varkappa^2 \end{array}\right\|, \quad \left\|\begin{array}{cccc} \lambda_r-\varkappa^2 & 1 & \ldots & 0 \\ 0 & \lambda_r-\varkappa^2 & \ldots & 0 \\ \ldots & & \ldots & \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & \lambda_r-\varkappa^2 \end{array}\right\|$$

where $\varkappa^{2k}$ is an elementary divisor of the first matrix. The elementary divisors of the second matrix are obtained after a decomposition of the polynomial $(\lambda_r-\varkappa^2)^{p_r}$ into factors irreducible in the complex number field. If $\lambda_r \in \sigma(Q)$ and $\lambda_r \notin \Theta$, then a simple analysis indicates that among the elementary divisors we can find a corresponding root with positive real part of the equation $f(\varkappa)=0$. In this case the solution $x \equiv 0$ is unstable. If $\sigma(Q) \cap \Theta = \Theta$, then the decomposition of the polynomial into irreducible factors yields $(\varkappa+i\sqrt{-\lambda_r})^{p_r}$, $(\varkappa-i\sqrt{-\lambda_r})^{p_r}$.

4. The validity of the item 4 of the Theorem is obvious.

We study the particular cases of the action of forces of various types on a scleronomous system.

Gyroscopic forces of the form $Q_i = \gamma_{i1}q_1 + \ldots + \gamma_{in}q_n$. The matrix $\Gamma = \|\gamma_{ij}\|_1^n$ is necessarily skew-symmetric. For system (1.2), $Q=0, L=A^{-1}\Gamma$. It is proved that $\sigma(A^{-1}\Gamma) \in \Omega$. The scalar product of vectors is defined by the formula $u \cdot v = u_1\bar{v}_1 + \ldots + u_n\bar{v}_n$ (the overbar denotes the complex conjugate); $\Gamma u \cdot u + \Gamma\bar{u}\cdot\bar{u} = 0$ for any u because $\Gamma = -\Gamma'$. If u is an eigenvector of the matrix $A^{-1}\Gamma$, corresponding to an eigenvalue λ, then $\Gamma u = \lambda A u$. Since $\lambda A u \cdot u + \bar{\lambda} A\bar{u}\cdot\bar{u} = 0$ and $Au\cdot u = A\bar{u}\cdot\bar{u} \neq 0$, we have $\lambda + \bar{\lambda} = 0$. On the basis of item 2 of Theorem 1, $\sigma(P) \in \Omega$. The matrix $A^{-1}\Gamma$ must be skew-symmetric for the matrices A^{-1} and Γ to commute. It possesses linear elementary divisors in the complex number field. The elementary divisors of the matrix $P - \varkappa E$ are of the types $\varkappa$, $\varkappa^2$, $(\varkappa + i\alpha)$, $(\varkappa - i\alpha)$ $(\alpha > 0)$. If, moreover, $\det \Gamma \neq 0$, then elementary divisors of the types $\varkappa$, $(\varkappa + i\alpha)$, $(\varkappa - i\alpha)$ correspond to the matrix P.

Dissipative forces $Q_i = -(b_{i1}q_1^{\cdot} + \ldots + b_{in}q_n^{\cdot})$, $B = \|b_{ij}\|_1^n \geqslant 0$. Here $Q=0$, $L=-A^{-1}B$. We have $\sigma(-A^{-1}B) \in \Theta$. Indeed, if u is an eigenvector of the matrix $-A^{-1}B$, corresponding to an eigenvalue λ, then

$$\lambda = -Bu\cdot u / Au\cdot u$$

We obtain what is required since $Au\cdot u > 0$, $Bu \cdot u \geqslant 0$. The spectrum $\sigma(P)$ consists of negative numbers and zeros. The matrix $(-A^{-1}B)$ must be symmetric for the matrices A^{-1} and B to commute. It possesses linear elementary divisors in the complex number field. Elementary divisors $\varkappa$, $\varkappa^2$, $(\varkappa+\alpha)$ $(\alpha > 0)$ correspond to the matrix P. If, moreover, $\det B \neq 0$, then the elementary divisors of $P - \varkappa E$ are simple.

Potential forces $Q_i = -\partial\Pi / \partial q_i$, where

$$\Pi = \frac{1}{2}\sum_{i,j}^{n} b_{ij}q_iq_j, \qquad B \geqslant 0$$

The system is conservative. Here $Q = -A^{-1}B$, $L = 0$. The proof of the algebraic fact $\sigma(-A^{-1}B) \in \Theta$ is obtained also from mechanical considerations. We select analytic functions $\psi(q)$ (of not lower than third order) such that the potential energy $\Pi + \psi(q)$ reaches a strict minimum when $q = 0$. We obtain what is required by using Lagrange's stability theorem and item 3 of Theorem 1. We can assert that in case A^{-1} and B commute and det $B \neq 0$ linear elementary divisors of the types $(\varkappa + i\alpha)$, $(\varkappa - i\alpha)$ correspond to the matrix P.

2. In the system of Eqs. (1.2) we assume $Q = 0$, $\nu(x, 0) \equiv 0$. Then (1.2) admits of the solution

$$x \equiv c, \qquad y \equiv 0 \tag{2.1}$$

where c is a constant vector. The vector c is said to be admissible if its Euclidean norm $|c|$ is sufficiently small. For system (1.2),

$$\nu(x, y) = Y(x)y + \nu^*(x, y)$$

The components of the vector $\nu^*(x, y)$ are of not less than second order in y and $Y(0) = L$.

T h e o r e m 2. If $Q = 0$, $\nu(x, 0) \equiv 0$, and the matrix $Y(c) = \|y_{ij}(c)\|^n{}_1$ is a Hurwitz matrix, then the solution (2.1) is Liapunov-stable.

P r o o f. Let $\mu_i(y)$ denote linear forms satisfying the equations

$$\sum_{j=1}^{n} [y_{j1}(c)y_1 + \ldots + y_{jn}(c)y_n] \frac{\partial \mu_i}{\partial y_j} = y_i \qquad (i = 1, \ldots, n) \tag{2.2}$$

System (2.2) is solvable because det $Y(c) \neq 0$. After the change of variables $x = \zeta + \mu(y) + c$, the initial system becomes

$$d\zeta / dt = \xi(\zeta, y), \qquad dy / dt = Y(c)\ y + \nu^c(\zeta, y) \tag{2.3}$$

$$\nu^c(\zeta, y) = \nu^*(\zeta + \mu(y) + c, y) + [Y(\zeta + \mu(y) + c) - Y(c)]y$$

$$\xi(\zeta, y) = -\sum_{j=1}^{n} \nu_j^0(\zeta, y) \frac{\partial \mu(y)}{\partial y_j}$$

The vectors $\nu^c(\zeta, y)$, $\xi(\zeta, y)$ are of not less than second order in ζ, y. The trivial solution of system (2.3) is stable [1] because $\xi(\zeta, 0) \equiv 0$, $\nu^c(\zeta, 0) \equiv 0$ and $Y(c)$ is a Hurwitz matrix.

The stability theorem for the trivial solution can be formulated also for the more general system of equations

$$\frac{d\xi}{dt} = \xi(x, y), \qquad \xi(0, 0) = 0, \ \xi = \left\| \begin{matrix} x \\ y \end{matrix} \right\| \tag{2.4}$$

where x and y are s and n-dimensional vectors. The components of the vector $\xi(x, y)$ are analytic in x, y; $\xi(x, 0) \equiv 0$. System (2.4) admits of solution (2.1). If solution (2.1) is stable, then for any (small) $\varepsilon > 0$ we can find a number set $X_\varepsilon(c)$ possessing the property: let $\alpha \in X_\varepsilon(c)$; from $|x(0) - c| < \alpha$, $|y(0)| < \alpha$ follows $|x(t) - c| < \varepsilon$, $|y(t)| < \varepsilon$ $(t \geqslant 0)$. The set $X_\varepsilon(c)$ is contained on the segment $[0, \varepsilon]$; $S(x^\circ, h) = \{x: |x - x^\circ| = h\}$ is a sphere of radius h with center at x°.

L e m m a. Let solution (2.1) be stable for any admissible c. Then, for sufficiently small h, ε we can find a number $\beta > 0$ such that

$$\inf_{x \in S(0, h)} \sup X_\varepsilon(x) > \beta$$

Proof. We assume the contrary. Then there exists a sequence $\{x^e\}$ $(x^e \in S(0, h))$, such that $\lim [\sup X_\varepsilon(x^e)] = 0$. The set $S(0, h)$ is closed (in the Euclidean metric), therefore, $\lim x^e = x^* \in S(0, h)$ as $e \to \infty$. The solution $x \equiv x^*$, $y \equiv 0$ is stable; for $\varepsilon > 0$ we can find $\gamma > 0$ smaller than ε, such that from

$$|x(0) - x^*| < \gamma, \qquad |y(0)| < \gamma$$

follows

$$|x(t) - x^*| < \varepsilon, \qquad |y(t)| < \varepsilon \qquad (\text{for } t \geqslant 0)$$

In its own turn, for γ we can find a number $\eta > 0$ such that from

$$|x(0) - x^*| < \eta, \qquad |y(0)| < \eta$$

follows

$$|x(t) - x^*| < \gamma, \qquad |y(t)| < \gamma \qquad (\text{for } t \geqslant 0)$$

By choosing the number N sufficiently large we can ensure the fulfillment of the relations

$$|x^e - x^*| < \eta/2, \qquad S(x^e, \eta/2) \subset S(x^*, \eta)$$

$$S(x^*, \gamma) \subset S(x^e, \varepsilon), \qquad e \geqslant N$$

Therefore, for any $e \geqslant N$ from

$$|x(0) - x^e| < \eta/2, \qquad |y(0)| < \eta/2$$

follows $|x(t) - x^e| < \varepsilon$, $|y(t)| < \varepsilon$ (for $t \geqslant 0$), i.e., $\sup X_\varepsilon(x^e) \geqslant \eta/2$ for $e \geqslant N$. The contradiction proves the lemma.

Theorem 3. Let solution (2.1) be stable for all admissible c ; let there exist a y-positive-definite function $V(y)$ such that $V^{\cdot}_{(2.4)} \leqslant 0$. Then the trivial solution of system (2.4) is stable.

Proof. Let

$$\beta = \inf_{x \in S(0, \varepsilon/2)} \sup X_{\varepsilon/2}(x)$$

On the basis of the lemma, $\beta \neq 0$. For β we can choose $\delta > 0$ such that $|y(t)| < \beta$ follows from the condition

$$|x(0)| < \delta, \qquad |y(0)| < \delta$$

for all $t \geqslant 0$ for which $|x(t)| < \varepsilon/2$. The possibility of choosing δ is stipulated by the sign-definiteness of $V(y)$ and the negativeness of $V^{\cdot}_{(2.4)}$ (for all x from a sufficiently small neighborhood of zero).

Therefore, even if the representative point leaves the sphere $S(0, \varepsilon/2)$, it does so only owing to the x coordinate. But then for some t^* we have $|x(t^*)| = \varepsilon/2$ and $|y(t^*)| < \beta$. The solution $x \equiv x(t^*)$, $y \equiv 0$ is stable. From the meaning of the number β follows $|x(t)| < \varepsilon$, $|y(t)| < \varepsilon/2 < \varepsilon$ for $t \geqslant t^*$.

Theorem 3 must be applied to the study of the stability of the trivial solution of the system of equations

$$dx/dt = y + \chi(x, y, z), \qquad dy/dt = \nu(x, y, z)$$

$$dz/dt = Gz + \zeta(x, y, z) \tag{2.5}$$

where x, y, $\chi(x, y, z)$, $\nu(x, y, z)$ are s-dimensional vectors and z, $\zeta(x, y, z)$ are n-dimensional vectors; G is a Hurwitz matrix. Liapunov had made a detailed investigation of (2.5) for $s = 1$. We can easily point out examples of matrices, equivalent to matrices of type (1.3), in the class of first-approximation matrices of system (2.5). Certain of Liapunov's results were carried over in [8 - 10] to the case $s > 1$ under the assumption $v(x, 0, 0) \equiv 0$. Without loss of generality, $\chi(x, 0, 0) \equiv 0$, $\zeta_*(x, 0, 0) \equiv 0$, $v(x, 0, z) \equiv 0$. System (2.5) admits of the solution $x \equiv c$, $y \equiv 0$, $z \equiv 0$. Under certain assumptions theorems analogous to Theorems 2 and 3 can be formulated for (2.5).

The author thanks V.V. Rumiantsev for valuable advice.

BIBLIOGRAPHY

1. Liapunov, A.M., General Problem of the Stability of Motion. Moscow-Leningrad, Gostekhizdat, 1950.
2. Liapunov, A.M., Investigation of One of the Singular Cases in the Problem of Stability of Motion. Leningrad, Izd. LGU, 1963.
3. Rumiantsev, V.V., The method of Liapunov functions in the stability theory of motion. In: Fifty Years of Mechanics in the USSR, Vol.1. Moscow, "Nauka" 1968.
4. Arnol'd, V.I., The algebraic unsolvability of the stability problem and of the problem of the topological classification of the singular points of systems of differential equations. Uspekhi Mat.Nauk, Vol.25, №2, 1970.
5. Gantmakher, F.R., Lectures on Analytical Mechanics. Moscow, "Nauka",1966.
6. Chetaev, N.G., Stability of Motion. Moscow, "Nauka", 1965.
7. Gantmakher, F.R., Theory of Matrices. Moscow, "Nauka", 1966.
8. Sagitov, M.S. and Filatov, A.N., On Liapunov-stability in the critical case of a characteristic equation with an even number of roots equal to zero. PMM Vol.29, №1, 1965.
9. Ngo Van Vyong., On the stability of motion in a certain critical case. PMM Vol.30, №4, 1966.
10. Veissenberg, A.N., On a transformation of a system of differential equations of perturbed motion in the critical case with an even number of zero roots. Izv. Vuzov, Matematika, №11, 1968.

Translated by N.H.C.